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Random fields of water surface waves using Wiener–Hermite functional series expansions

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Random motions of irrotational gravity water surface waves on deep water are formulated using the so-called Wiener–Hermite functional series expansion, based on the 'ideal random process', i.e. the white noise. Such a procedure is known to differ fundamentally from moment expansions such as Gram–Charlier or Edgeworth series. The applications concern 'free waves' which are homogeneous in the horizontal plane and stationary in time. Starting from the basic hydrodynamic equation and boundary conditions, the general procedure for obtaining the equations for the deterministic kernels is described. First, the expansion is carried out with no approximation of the hydrodynamic equations but the expansion is limited to the first order. This defines the Gaussian part of the wave field. As expected, the nonlinearity of the hydrodynamic equations has effects on the dispersion relation through explicit frequency and acceleration terms whose physical interpretations are discussed. No attempt is made to solve the highly complicated coupled nonlinear integral kernels equations. Instead, Dirac kernel functions are chosen *à priori* as an approach to a narrowband random wave field. In this case, the nonlinearity is found to be characterized by a 'statistical wave steepness' having an upper limit value of order 0.42. As a second example, a non-Gaussian field is determined on the basis of the hydrodynamic equations truncated at second order in the wave amplitude. In the case of Dirac first-order kernels, the second-order nonlinear effect results in the generation of the second harmonic of the fundamental wave component. The ratio between the energy levels of these two components is found to compare well with standard results from laboratory experiments.

1. Introduction

During the past few decades, using theoretical, experimental as well as numerical approaches, considerable progress has been made in the analysis of deterministic water surface waves. For irrotational waves, the mathematical problems to be solved are known to be well-posed, but difficulties rapidly arise in trying to find solutions especially when the nonlinear terms are to be taken into account. The use of new concepts and techniques, mostly numerical, allow to now be overcome many of the difficulties.

Progress in the analysis of random wave motions is clearly much less important and results from the deterministic case were often applied to such motions without clear mathematical and physical justification. As is well-known, a complete characterization

of a random field requires the determination of multidimensional distribution functions of the physical variables involved (see e.g. Sobczyk 1984). As in the case of many geophysical phenomena, such determination seems still out of the reach of theoretical as well as experimental investigation on surface waves and the characterization is often limited to few quantities through basic assumptions such as homogeneity in space and/or stationarity in time. Among the standard models of random water surface waves, those of Eckart (1953), Phillips (1957), Tick (1959, 1963) and West (1982) are of particular interest here. The first two authors used the linearized hydrodynamic equations to study the response of the water surface to random perturbations (pressure) in the air flow. The main objective was to explain the first stage of wave generation by the wind, the water surface being initially at rest. The results are presented in terms of the time evolution of the wave spectra, with the assumption that the wave fields under study are homogeneous in horizontal space.

Being more interested in the nonlinear effects, we look into more detail at the works of the last two authors. Tick (1959) used the perturbation technique around the water surface level at rest to determine the second-order correction to the solution of the linearized forms of the hydrodynamic equations. The wave field associated with these forms is assumed to be Gaussian. Then, the second-order correction is expected to, in particular, explain the indication from observations that a small secondary peak appears in the wave spectrum at about twice the frequency of the primary peak. From the mathematical viewpoint, as pointed out by Tick, a double perturbation is then performed, first on the hydrodynamic equations, about some equilibrium condition and second on the probability structure, about the Gaussian 'point'. In a subsequent paper (Tick 1963) he concluded that the second-order correction to the first-order wave spectrum is small and raised question about the validity of his method. The model of West (1982) is conceptually different. It is mostly related to the field of a nonlinear stochastic dynamical system. The water surface is seen as an ensemble of nonlinear dissipative oscillators under the effect of external random forcing. The nonlinearity involves the wave–wave interactions, the dissipation is due to viscosity, while the wind provides the external forcing. Thus, the model includes all of the main processes generally involved in the evolution of a wind wave field. Through the use of so-called stochastic linearization, the Langevin equation which governs the normal mode amplitudes is derived. The flux from the wind being assumed a Gaussian delta-correlated process, a Fokker–Planck equation governs the evolution of the probability density function of the wave system. This function and thus the steady-state energy spectral density are determined explicitly.

While the Phillips model constitutes a basic reference in the field of wind waves, the other models are much less cited in the current literature. Apart from the preliminary attempts of their authors, no detailed comparison of the results with observations have been reported and the real importance of these models is still not known. Note that, in some of these models, an appeal to the results or to mathematical techniques from deterministic analysis are made. This may raise questions about their classification as random models.

Our approach uses a different technique to determine the random wave fields. The technique is now known as the Wiener–Hermite functional series expansion. It follows the work of Wiener (1939) who proposed studying hydrodynamic turbulence by introducing expansions of the physical variables in terms of the white-noise random process. Extensive applications in this specific context are due to Meecham and his co-workers (see e.g. Meecham & Siegel 1964; Siegel, Imamura & Meecham 1965; Meecham & Jeng 1968; Lee, Meecham & Hogge 1982). Once the early works of these

authors were published, Saffman (1968) declared that the Wiener–Hermite expansion "gives hope that a satisfactory analytical description of homogeneous turbulence may be obtained fairly soon". Limitations of the method in accounting for the nonlinear energy cascade were analysed by Crow & Canavan (1970).

More recently, the study of fully developed turbulence was treated with this method by Meecham (1999). Ahmadi (1980, 1982) introduced the method for analysing earthquake and plasma turbulence. Then, the method found important applications in the fields of nonlinear stochastic dynamic systems and electromagnetic wave propagation (Jahedi & Ahmadi 1980, 1983; Orabi & Ahmadi 1987; Eftimiu 1989; Eftimiu & Pan 1990; Ogura 1995; Skaropoulos & Chrissoulidis 1999; Meecham & Lin 2001). An application to quantum dynamics with stochastic energy fluctuation was also carried out by Kayanuma & Noba (2001).

Recent more theoretical developments are now available to extend the potential of the original Wiener expansion method to new applications (see for example Jardak & Ghanem 2002; Xui *et al.* 2002). Concerning the ability of and the interest in the method to treat ocean dynamics, the National Research Council (1994) announced that "simulated non-Gaussian random (ocean) field that satisfy basic conservation laws of fluid dynamic represent great interest. A possible way of constructing individual realizations of a random field might be via the use of Wiener–Hermite polynomials (i.e. Wiener–Ito expansion)... the Wiener–Ito expansion has never been used, although it appears to be most relevant". To our knowledge, as far as surface waves are concerned, the present work constitute the first attempt to apply the Wiener–Hermite technique. A very brief presentation of the work, limited to the Gaussian case can be found in Joelson & Ramamonjiarisoa (1999).

Before ending this brief review of various works, it is appropriate to mention the weak wave–turbulence theory, extensive applications of which have been made in the study of water surface waves since the fundamental work of Zakharov (1968). The applications yielded striking progress in the understanding of this field including wind-driven waves (see e.g. Newell & Zakharov 1992). They are mostly concerned with the evolution of the wave fields under the influence of weak nonlinear wave–wave interactions. Strictly speaking, they do not belong to the class of analyses that we reported above. Indeed, their main objective is to predict the evolution of the wave fields spectra (Zakharov & Filonenko 1966; Pushkarev & Zakharov 1998). In the terminology of statistical theory, they constitute second-order moment approaches. On the other hand, as stressed previously, the main aim of the Wiener–Hermite expansion method is to determine the wave fields probabilistic structures. Clearly, these two objectives differ fundamentally from each other. According to the basic principles, if needed, second-order moments (correlations and spectra) can be derived from the results of the Wiener–Hermite analysis. This will be explicitly seen in the examples which follow.

This paper is organized as follows: in § 2, we will recall the main properties of the Wiener–Hermite functional expansion. Then, in § 3, the application to the basic complete equations governing surface wave fields on deep water is presented, assuming homogeneity and stationarity. Then, the kernels equations corresponding to the Gaussian part of the fields are derived and examined in § 4. In § 5, the specific case of Dirac forms of the kernels is treated to illustrate the mathematical analysis. A second application, of particular practical interest, is concerned with the hydrodynamic equations truncated at second order (§ 6). The application then includes the double approximation of Tick (1959) as defined previously. The associated wave field now departs from a Gaussian field due to the second-order nonlinearity. The

amount of this departure is measured and compared with some experimental results. Finally, a discussion of the results follows and possible extensions of the work are proposed.

2. Basic aspects of the Wiener–Hermite functional expansion

We recall briefly here some fundamental aspects of the Wiener–Hermite functional expansion. Details can be found in many textbooks and articles cited in the references. The Wiener–Hermite series expansion method was developed by Wiener (1958) by extending to random functions the expansion method of Cameron & Martin (1947). It constitutes a mathematical procedure, which allows an arbitrary random function to be represented in terms of a complete set (the Wiener–Hermite set) of elementary random processes.

The zeroth-order member of the set is unity and the first-order member is the so-called 'white noise', $a(t, \alpha)$, that is the derivative of the Wiener process $X(t, \alpha)$ (Wiener 1958):

$$H^{(0)} = 1, \tag{2.1}$$

$$H^{(1)} = a(t, \alpha), \tag{2.2}$$

with

$$a(t, \alpha) = \frac{\mathrm{d}}{\mathrm{d}t} X(t, \alpha), \tag{2.3}$$

t being the physical variable (time) and α the 'event' variable. It has zero mean and is δ-correlated, namely

$$\langle a(t, \alpha) \rangle = 0, \tag{2.4}$$

$$\langle a(t_1, \alpha) a(t_2, \alpha) \rangle = \delta(t_1 - t_2), \tag{2.5}$$

where the angular brackets denote an ensemble average on the α-set. For convenience, the variable will be omitted in what follows. Then, the higher members, $H^{(j)}$, of the set are specified in terms of $H^{(1)}$.

In this work, the expansions will be limited, at most, to the second order. The second member of the set is defined as

$$H^{(2)}(t_1, t_2) = H^{(1)}(t_1) H^{(1)}(t_2) - \delta(t_1 - t_2). \tag{2.6}$$

From the definitions, it follows that, except for the zeroth-order member, all members of the Wiener set are random functions with zero mean, i.e.

$$\left\langle H^{(j)}(t_1, t_2, \ldots, t_j) \right\rangle = 0. \tag{2.7}$$

Also, the Wiener set is an orthogonal set, in a statistical sense, i.e.

$$\left\langle H^{(i)}(t_1, t_2, \ldots, t_i) H^{(j)}(t_1, t_2, \ldots, t_j) \right\rangle = 0 \qquad \text{if} \quad i \neq j. \tag{2.8}$$

Figure 1 illustrates 'physical realizations' of $H^{(1)}$ and $H^{(2)}$ by using standard random number generators. $H^{(1)}$ exhibits the classical form of a Gaussian white noise.

The Wiener–Hermite set is complete in the sense that, as shown by Wiener (1958), any arbitrary random function $f(t)$ of ('time') t, with zero mean can be expanded in the following functional form:

$$f(t) = \sum_{n=1}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_n(t, t_1, t_2, \ldots, t_n) H^{(n)}(t_1, t_2, \ldots, t_n) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \ldots \mathrm{d}t_n. \tag{2.9}$$

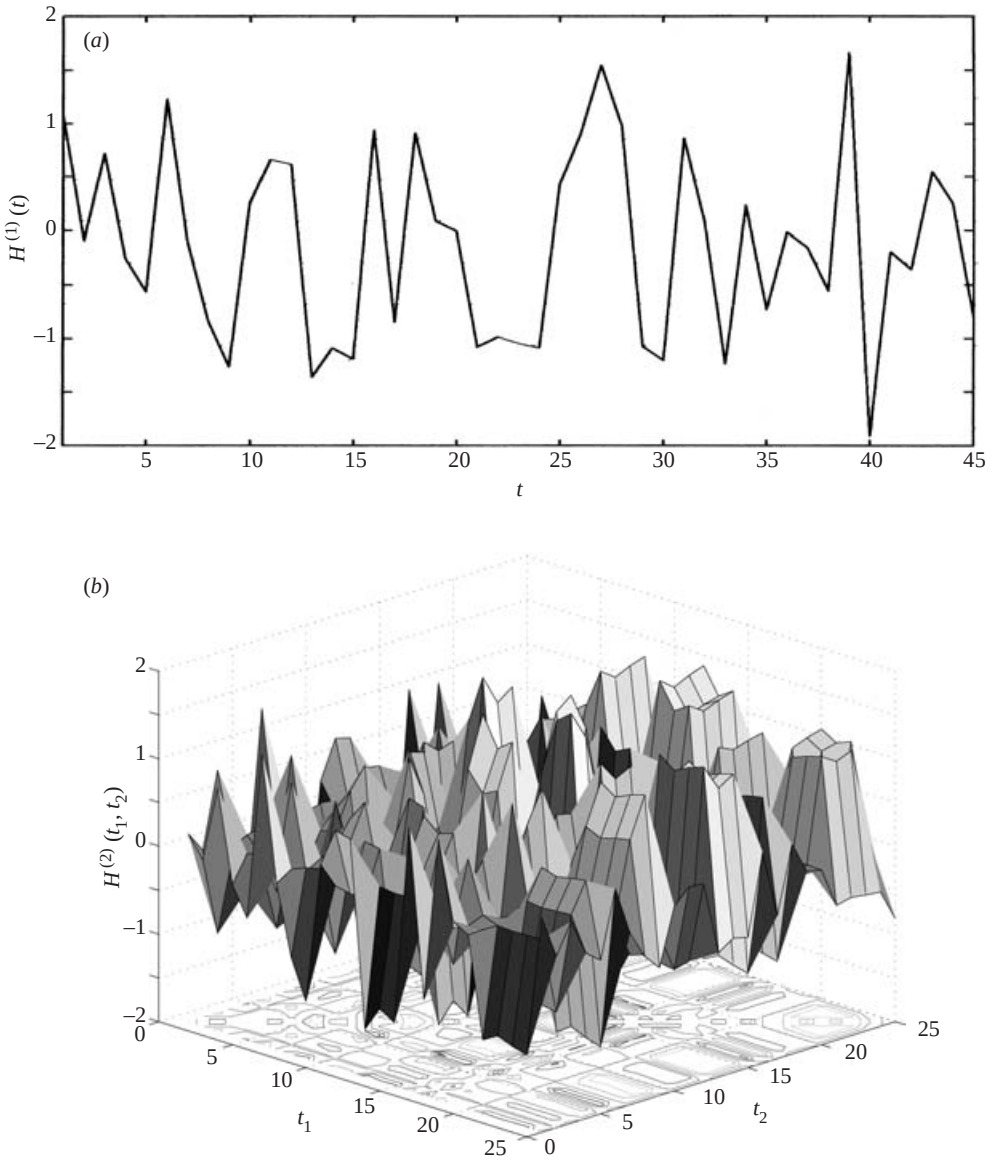

FIGURE 1. Short samples of normalized (zero mean, unity standard deviation) 'time' evolutions of (*a*) $H^{(1)}(t)$, (*b*) $H^{(2)}(t_1, t_2)$, with the contour levels added.

This expansion converges to the original random function with probability one.

$F_1, F_2, \ldots, F_n$ are respectively the first, the second and the nth order kernels. It is of fundamental importance to note that these kernels are deterministic functions of the arguments. Thereby, they represent mathematically the projection of the random function on the Wiener–Hermite set. Clearly, if the expansion is limited to the first term, $f(t)$ will be Gaussian. Then, the higher-order terms represent the departure from a normal process. The case of a stationary random function is of particular interest in many applications In this case, the kernels are function only of the differences of the arguments, that are $t - t_1$ for F_1, $t - t_1$ and $t - t_2$ for F_2 so on. In the applications,

the expansion is generally limited to a finite number of terms. Then, $f(t)$ is written as

$$f(t) = \int_{-\infty}^{+\infty} F_1(t-t_1)H^{(1)}(t_1)\,\mathrm{d}t_1 + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_2(t-t_1, t-t_2)H^{(2)}(t_1, t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 + \cdots. \tag{2.10}$$

In this case, it appears convenient to apply the expansion in the Fourier space rather than in the physical space. This is done through the usual Fourier–Stieltjes transformation (see e.g. Bochner 1960):

$$f(t) = \int_{-\infty}^{+\infty} \mathrm{d}A(\omega)\exp(\mathrm{i}\omega t). \tag{2.11}$$

The Wiener–Hermite expansion of the (complex) random amplitude corresponding to (2.10) then writen (see e.g. Joelson 1997)

$$\mathrm{d}A(\omega) = A_1(\omega)\tilde{H}^{(1)}(\omega) + \int_{-\infty}^{+\infty} A_2(\omega_1, \omega-\omega_1)\tilde{H}^{(2)}(\omega_1, \omega-\omega_1)\,\mathrm{d}\omega_1 + \cdots. \tag{2.12}$$

Again, the kernels $A_1, A_2, \ldots$ are deterministic functions of the arguments. They are the Fourier transform of $F_1, F_2, \ldots$, while $\tilde{H}^{(1)}, \tilde{H}^{(2)}, \ldots$, are the Fourier–Stieltjes transform of $H^{(1)}, H^{(2)}, \ldots$. Note that the set $\tilde{H}^{(j)}$ maintains the orthogonality property of the original set.

More generally, given a random function $f(\boldsymbol{x}, t)$, homogeneous in the horizontal plane ($\boldsymbol{x}$) and stationary in time, the Fourier–Stieltjes representation is writen

$$f(\boldsymbol{x}, t) = \int_{\boldsymbol{k}}\int_{\omega} \mathrm{d}A(\boldsymbol{k}, \omega)\exp\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t). \tag{2.13}$$

The corresponding Wiener–Hermite expansion of the complex random amplitude is

$$\begin{aligned}\mathrm{d}A(\boldsymbol{k}, \omega) &= A_1(\boldsymbol{k}, \omega)\tilde{H}^{(1)}(\boldsymbol{k}, \omega)\\ &+ \int_{\boldsymbol{k}_1}\int_{\omega_1} A_2(\boldsymbol{k}_1, \omega_1, \boldsymbol{k}-\boldsymbol{k}_1, \omega-\omega_1)\tilde{H}^{(2)}(\boldsymbol{k}_1, \omega_1, \boldsymbol{k}-\boldsymbol{k}_1, \omega-\omega_1)\,\mathrm{d}\omega_1\,\mathrm{d}\boldsymbol{k}_1 + \cdots.\end{aligned} \tag{2.14}$$

In the applications which follow, it is more appropriate to determine the kernels in the Fourier space rather than in the physical space. Indeed, such a procedure can greatly help in interpreting the results on physical grounds.

3. Applications to water surface waves

The water surface wave field of interest is assumed to be governed by the usual equation and boundary conditions for irrotational motion on deep water (see e.g. Phillips 1960). Let t, $\boldsymbol{x} = (x, y)$, and z be respectively the time, the horizontal coordinates and the vertical coordinate, g, the acceleration due to gravity, ϕ the velocity potential and η the water surface elevation. The equations are the Laplace equation

$$\frac{\partial^2}{\partial x^2}\phi(\boldsymbol{x}, z, t) + \frac{\partial^2}{\partial y^2}\phi(\boldsymbol{x}, z, t) + \frac{\partial^2}{\partial z^2}\phi(\boldsymbol{x}, z, t) = 0; \tag{3.1}$$

the kinematic boundary condition

$$\frac{\partial}{\partial t}\eta(\boldsymbol{x}, t) + \nabla_h\phi(\boldsymbol{x}, z, t)\Big|_{z=\eta(\boldsymbol{x}, t)} \cdot \nabla_h\eta(\boldsymbol{x}, t) - \frac{\partial}{\partial z}\phi(\boldsymbol{x}, z, t)\Big|_{z=\eta(\boldsymbol{x}, t)} = 0; \tag{3.2}$$

the dynamic boundary condition

$$g\eta(\boldsymbol{x},t)+\frac{\partial}{\partial t}\phi(\boldsymbol{x},z,t)\bigg|_{z=\eta(\boldsymbol{x},t)}+\tfrac{1}{2}(\nabla\phi(\boldsymbol{x},z,t))^2\big|_{z=\eta(\boldsymbol{x},t)}=0; \tag{3.3}$$

and the radiation condition

$$\lim_{z\to-\infty}\phi(\boldsymbol{x},z,t)=0. \tag{3.4}$$

The random wave fields will now be formulated on the basis of these equations through the harmonic decomposition and the Wiener–Hermite functional expansions of the physical variables. Assuming a wave field, homogeneous in the horizontal plane and stationary in time, the physical variables of interest can be written as

$$\eta(\boldsymbol{x},t)=\int_{\boldsymbol{k}}\int_{\omega}\mathrm{d}N(\boldsymbol{k},\omega)\exp\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t), \tag{3.5}$$

$$\phi(\boldsymbol{x},z,t)=\int_{\boldsymbol{k}}\int_{\omega}\mathrm{d}B'(\boldsymbol{k},\omega,z)\exp\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t), \tag{3.6}$$

where $\mathrm{d}B'(\boldsymbol{k},\omega,z)$ is the product of a deterministic function $f(z)$ and a random amplitude $\mathrm{d}B(\boldsymbol{k},\omega)$, namely,

$$\mathrm{d}B'(\boldsymbol{k},\omega,z)=\mathrm{d}B(\boldsymbol{k},\omega)f(z). \tag{3.7}$$

From the Laplace equation and the radiation condition one obtains the standard solution $f(z)=\exp(kz)$ with $k=|\boldsymbol{k}|$.

An expansion series of the exponential function yields

$$\phi(\boldsymbol{x},z,t)=\int_{\boldsymbol{k}}\int_{\omega}\mathrm{d}B(\boldsymbol{k},\omega)\sum_{n=0}^{\infty}\frac{(|\boldsymbol{k}|z)^n}{n!}\exp\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t). \tag{3.8}$$

Introducing this expansion in the boundary conditions (3.2) and (3.3), very long and very tedious algebraic operations (for details, see Joelson 1997) yield finally the following expressions:

the kinematic boundary condition

$$|\boldsymbol{k}|\,\mathrm{d}B(\boldsymbol{k},\omega)+\mathrm{i}\omega\,\mathrm{d}N(\boldsymbol{k},\omega)=-\sum_{n=1}^{\infty}\int_{\boldsymbol{k}_1}\int_{\omega_1}\cdots\int_{\boldsymbol{k}_n}\int_{\omega_n}\frac{\left|\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right|^{(n-1)}}{(n-1)!}$$

$$\times\left[\frac{\left|\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right|^2}{n!}+\left(\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right)\cdot\boldsymbol{k}_1\right]\mathrm{d}N(\boldsymbol{k}_1,\omega_1)\ldots\mathrm{d}N(\boldsymbol{k}_n,\omega_n)\,\mathrm{d}B\left(\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j,\omega-\sum_{j=1}^{n}\omega_j\right); \tag{3.9}$$

the dynamic boundary condition

$$\mathrm{i}\omega\,\mathrm{d}B(\boldsymbol{k},\omega)-g\,\mathrm{d}N(\boldsymbol{k},\omega)=\sum_{n=1}^{\infty}\int_{\boldsymbol{k}_1}\int_{\omega_1}\cdots\int_{\boldsymbol{k}_n}\int_{\omega_n}\left[-\mathrm{i}\frac{\left(\omega-\sum_{j=1}^{n}\omega_j\right)\left|\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right|^{n}}{n!}\,\mathrm{d}N(\boldsymbol{k}_1,\omega_1)\right.$$

$$\left.+\left(\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right)\cdot\boldsymbol{k}_1\cdot\frac{\left(\left|\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j\right|+|\boldsymbol{k}_1|\right)^{(n-1)}}{(n-1)!}\,\mathrm{d}B(\boldsymbol{k}_1,\omega_1)\right]\mathrm{d}N(\boldsymbol{k}_2,\omega_2)\ldots\mathrm{d}N(\boldsymbol{k}_n,\omega_n)$$

$$\times\,\mathrm{d}B\left(\boldsymbol{k}-\sum_{j=1}^{n}\boldsymbol{k}_j,\omega-\sum_{j=1}^{n}\omega_j\right)=0. \quad (3.10)$$

It may be of interest to note that the terms in the integrals in equations (3.9) and (3.10) have two different origins: the first terms correspond to the linear forms $(\partial/\partial z)\phi$ and $(\partial/\partial t)\phi$ at the surface level while the second terms are associated with the strictly nonlinear terms in the respective boundary conditions. The two types of terms differ from each other in that the first type involves differences of frequency or wavenumbers while the second type involves products of wavenumbers.

Note that the procedure above has already been carried out by Phillips (1960) but for an homogeneous and non-stationary field. In addition, the expansion of the dynamic boundary condition was truncated at a certain order. The next step is to expand the random amplitudes in terms of the Wiener–Hermite set. The nature of these amplitudes will be defined by the extent of the expansions. More precisely, if the expansions are limited to their respective first terms, only the Gaussian part of the wave fields will be determined. If higher-order terms are introduced, the defined wave fields will depart from a Gaussian fields, the departure increasing with the order of the Wiener–Hermite functional expansion. Of course, the exact solution would include an infinite sum of terms in the expansion. Clearly, extreme algebraic difficulties can be anticipated in determining the solution. Thus, in practice, the expansion is limited to finite numbers (often to the first) of terms. In view of the fast convergence of the expansion (Wiener 1958), the approximate solution so obtained is expected to be a satisfactory representation of the exact solution (see e.g. Eftimiu & Pan 1990). Of most importance is that the fields of interest are known to be near-Gaussian. This would justify the use of truncated sums.

4. Gaussian part of the wave field

This Gaussian part will be determined by limiting the Wiener–Hermite expansion to the first term. Such procedure may a surprising at first sight seem in view of the nonlinearity of the governing equations. However, it is common in most applications of the expansion (see e.g. Ahmadi 1980; Eftimiu & Pan 1990; Orabi & Ahmadi 1987). Two types of arguments are used to justify the procedure:

(i) From the mathematical viewpoint, the Wiener–Hermite expansion, as given by expression (2.9) does not constitute a small perturbation procedure. Rather, it

represents an expression of a random function in terms of a set of elementary random processes built on the basis of the Gaussian white noise process. No ordering 'small parameter' is introduced. In practice, there is a need to truncate the expansion. But there are no consistency criteria to be satisfied relating the order of truncation to the form of equations which govern the process. Saffman (1968) pointed out that "the error involved in the truncation clearly cannot be assessed". However, we believe that the definition of a metric of the error in the Wiener–Hermite expansion is a central point in the estimation of the uncertainty introduced by the truncature order. As pointed out by one of the referees of this paper, it would be possible but it is clearly beyond the scope of the present paper. It is also emphasized that limiting the expansion to the first term does not constitute a linearization procedure. As shown by numerous examples in the literature and the analysis which follows, nonlinearity, clearly, has dominant effects.

(ii) Observational results and/or physical arguments suggest that in most applications, the random fields of interest are close to Gaussian. This is known to be the case for wind-driven waves (Kinsman 1965). Thus, it is expected that the first term in the Wiener–Hermite expansion contains the basic probabilistic features of the fields.

By limiting the amplitudes $\mathrm{d}N(\boldsymbol{k}, \omega)$ and $\mathrm{d}B(\boldsymbol{k}, \omega)$ to first order in the Wiener–Hermite expansion, one can write

$$\mathrm{d}N(\boldsymbol{k}, \omega) = N_1(\boldsymbol{k}, \omega) H^{(1)}(\boldsymbol{k}, \omega), \tag{4.1}$$

$$\mathrm{d}B(\boldsymbol{k}, \omega) = B_1(\boldsymbol{k}, \omega) H^{(1)}(\boldsymbol{k}, \omega). \tag{4.2}$$

The procedure to obtain the system of equations for the deterministic kernels consists of the substitution of (4.1) and (4.2) in the boundary conditions (3.9) and (3.10). The expressions so obtained are multiplied by $H^{(1)}(-\boldsymbol{k}, -\omega)$ and the ensemble average taken, accounting for the orthogonality property of the Wiener set. This involves very lengthy and complicated algebra and details can be found in Joelson (1997). We give here the final equations. The kinematic boundary condition (3.9) yields

$$k \exp\left(\tfrac{1}{2} k^2 \sigma_\eta^2\right) [1 + \nu_{1N}(\boldsymbol{k})] B_1(\boldsymbol{k}, \omega) + \mathrm{i}[\omega + \omega_c + \boldsymbol{k} \cdot \boldsymbol{V}_c] N_1(\boldsymbol{k}, \omega) = 0 \tag{4.3}$$

while the dynamic boundary condition (3.10) yields

$$-\mathrm{i} \exp\left(\tfrac{1}{2} k^2 \sigma_\eta^2\right) [\omega + \omega_d(\boldsymbol{k})] B_1(\boldsymbol{k}, \omega) + (g + \gamma_1 + \gamma_2 + \gamma_3) N_1(\boldsymbol{k}, \omega) = 0. \tag{4.4}$$

Although the above set of equations looks like a linear set, this is not the case as the various quantities appearing in these equations depend upon the unknown kernels as follows:

$$\nu_{1N}(\boldsymbol{k}) = \boldsymbol{k} \cdot \int_{\boldsymbol{k}'} \int_{\omega'} \boldsymbol{k}' N_1(\boldsymbol{k}', \omega') N_1^*(\boldsymbol{k}', \omega') \, \mathrm{d}\boldsymbol{k}' \omega' \tag{4.5}$$

and

$$\sigma_\eta^2 = \int_{\boldsymbol{k}'} \int_{\omega'} N_1(\boldsymbol{k}', \omega') N_1^*(\boldsymbol{k}', \omega') \, \mathrm{d}\boldsymbol{k}' \omega', \tag{4.6}$$

$$\omega_c = \mathrm{i} \int_{\boldsymbol{k}'} \int_{\omega'} \left\{ k'^3 \left[\exp\left(\frac{k'^2 \sigma_\eta^2}{2} \right) - 1 \right] \left[1 - \frac{1 + \epsilon^2(k')}{\epsilon^2(k')} \nu_{1N}(\boldsymbol{k}') - \nu_{1N}(\boldsymbol{k}') \exp\left(\frac{\epsilon^2(k')}{2} \right) \right] \right\} C_{NB}(\boldsymbol{k}', \omega') \, \mathrm{d}\boldsymbol{k}' \, \mathrm{d}\omega', \tag{4.7}$$

$$\boldsymbol{V}_c = -\mathrm{i}\int_{\boldsymbol{k}'}\int_{\omega'} \boldsymbol{k}' k' \exp\left(\frac{k'^2\sigma_\eta^2}{2}\right) C_{NB}(\boldsymbol{k}', \omega')\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega', \tag{4.8}$$

$$\omega_d(\boldsymbol{k}) = \mathrm{i}\int_{\boldsymbol{k}'}\int_{\omega'} (kk' + \boldsymbol{k}\cdot\boldsymbol{k}')(k'+k)\exp\left(\frac{(k'^2+2kk')\sigma_\eta^2}{2}\right) C_{NB}(\boldsymbol{k}', \omega')\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega', \tag{4.9}$$

$$\gamma_1 = \int_{\boldsymbol{k}'}\int_{\omega'} \mathrm{i}\omega' k'^2 \exp\left(\frac{k'^2\sigma_\eta^2}{2}\right) C_{NB}(\boldsymbol{k}', \omega')\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega', \tag{4.10}$$

$$\gamma_2 = \int_{\boldsymbol{k}'}\int_{\omega'} 2k'^3 \exp\left(2k'^2\sigma_\eta^2\right) \nu_B(\boldsymbol{k}', \omega')\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega', \tag{4.11}$$

$$\gamma_3 = -\frac{1}{2}\int_{\boldsymbol{k}'}\int_{\omega'}\int_{\boldsymbol{k}''}\int_{\omega''} (k'k'' + \boldsymbol{k}'\cdot\boldsymbol{k}'')(k'+k'')^3 \exp\left(\frac{(k'+k'')^2\sigma_\eta^2}{2}\right) \times C_{NB}(\boldsymbol{k}', \omega')C_{NB}(\boldsymbol{k}'', \omega'')\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}''\,\mathrm{d}\omega'', \tag{4.12}$$

with

$$C_{NB}(\boldsymbol{k}, \omega) = N_1(\boldsymbol{k}, \omega)B_1^*(\boldsymbol{k}, \omega), \tag{4.13}$$

$$\nu_B(\boldsymbol{k}, \omega) = B_1(\boldsymbol{k}, \omega)B_1^*(\boldsymbol{k}, \omega). \tag{4.14}$$

Among these quantities, some are standard statistical parameters: σ_η, $C_{NB}(\boldsymbol{k}, \omega)$ and $\nu_B(\boldsymbol{k}, \omega)$ are respectively the standard deviation of the water deflection level, the cross-spectrum of the velocity potential at the surface and the water deflection level, and the wavenumber–frequency spectrum of the velocity potential. The others quantities represent additional frequencies, velocity, accelerations, etc., related to the wave field nonlinearity. It is of interest to note that the nonlinear terms of the boundary conditions originally concentrated in unique terms in the physical space ((6.5) and (6.6)) or in infinite integrals over all wavenumbers and frequencies in the fourier space ((3.9) and (3.10)), have now split into these many different quantities whose physical interpretation will be attempted.

According to the previous remarks on the origins of the terms in the integrals in equations (3.9) and (3.10), the terms $k\exp(\frac{1}{2}k^2\sigma_\eta^2)B_1(\boldsymbol{k}, \omega)$ and $-\mathrm{i}\omega\exp(\frac{1}{2}k^2\sigma_\eta^2)B_1(\boldsymbol{k}, \omega)$ in equations (4.3) and (4.4) come from the linear forms $(\partial/\partial z)\phi$ and $(\partial/\partial t)\phi$ applied at the surface level.

We note that the quantities ω_c, $\boldsymbol{V}_c$, $\omega_c(\boldsymbol{k})$, γ_1 and γ_3 being real suggests that B_1 and N_1 are in quadrature. As a consequence, their cospectrum would be zero. This is indeed the case as proved by Joelson (1997). An illustration will be given herein for the particular case of a zero-bandwidth process.

The equations (4.3) and (4.4) constitute the system of coupled nonlinear integral equations for the kernels. The resolution of this system is clearly a considerable task and is left for future work. We will limit ourselves here to a preliminary attempt to physically interpret quantities which appear in the equations. It is to be recalled that they constitute averaged quantities. On the other hand, they are defined by integrals over wavenumber and frequency, and thus they constitute cumulative effects of elementary processes characterized by their respective integrands. The attempted physical interpretations will thus be concerned with these integrands.

We recall first that the kinematic boundary condition represents the fact that a fluid particle at the surface will stay at that surface during the wave motion. As far as the kinematic boundary condition is concerned, from equation (4.3), it is seen that for the Gaussian wave field, the nonlinearity of the condition yields the three quantities ν_{1N}, ω_c and $\boldsymbol{V}_c$.

Let $\boldsymbol{V}_c(\boldsymbol{k}', \omega')$ be the integrand of $\boldsymbol{V}_c$. It is seen that

$$\boldsymbol{V}_c(\boldsymbol{k}', \omega') \propto [k' N_1(\boldsymbol{k}', \omega')][k' B_1(\boldsymbol{k}', \omega')],$$

which represents the amplitude of the horizontal velocity at the surface times the dimensionless wave amplitude. The above product can be interpreted as the horizontal velocity at the surface weighted by the dimensionless wave amplitude. In view of the exponential terms in the integrand, the amplitude of $\boldsymbol{V}_c(\boldsymbol{k}', \omega')$ depends also upon the total variance of the wave field amplitude. Finally, $\boldsymbol{V}_c$ is the cumulative horizontal velocity at the surface associated with the ensemble of harmonic components of the field. Within the random nonlinear wave field, it would be the equivalent of the so-called Stokes drift velocity in a weakly nonlinear deterministic field. Consequently, $\boldsymbol{k} \cdot \boldsymbol{V}_c$ is the Doppler shift frequency associated with this averaged drift velocity.

The frequency ω_c can also be interpreted that way: it appears as the sum of the elementary Doppler shift frequencies due to the effects of the horizontal velocity at the surface related to each of the harmonic components of the corresponding wavenumber. From equation (4.3), $\nu_{1N}(\boldsymbol{k})$ represents the change in the wavenumber of the considered component due to the presence of the other components.

According to (4.4), the nonlinearity yields additional frequency and accelerations terms in the dynamic boundary condition:

(*a*) From (4.9), the frequency $\omega_d(\boldsymbol{k})$ results from the interactions between the considered mode and the other modes ($\boldsymbol{k}'$). It would be the equivalent of the frequency change in a weakly nonlinear deterministic wave field due to the interactions of two wave trains.

(*b*) From (4.10), γ_1 appears as a cumulative acceleration associated with the elementary vertical velocities $k'^2 \exp(\frac{1}{2}k'^2\sigma_\eta^2) C_{NB}(\boldsymbol{k}', \omega')$.

(*c*) From (4.12), the integrand of γ_3 can be written as

$$[\text{integrand of } \omega_d(\boldsymbol{k}'')] \exp\left(\tfrac{1}{2}k'^2\sigma_\eta^2\right)(k' + k'')^2 C_{NB}(\boldsymbol{k}', \omega').$$

This additional acceleration would be related to dual mutual interactions in the wave field: one mode changes the frequency of another mode. Within the vertical velocity field of the first mode, this change yields a vertical acceleration. In the above interpretations, a distinction is made between horizontal and vertical components of velocities and accelerations by noting that the horizontal component depends upon $\boldsymbol{k}$ while the vertical component depends upon k.

Before ending these preliminary comments, it is of importance to note that, due to the very rapid increase of the exponential terms, the convergence of the various integrals in the kernel equations requires these kernels to decrease rapidly with ω and $\boldsymbol{k}$.

Clearly, for the case of a homogeneous and stationary Gaussian field, the effect of the nonlinearity is purely kinematical. This is represented in terms of the dispersion relation obtained by combining (4.3) and (4.4). It is writen

$$k \exp\left(\tfrac{1}{2}k^2\sigma_\eta^2\right)[1 + \nu_{1N}(\boldsymbol{k})]\,(g + \gamma_1 + \gamma_2 + \gamma_3) = (\omega - \mathrm{i}\omega_c - \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{V}_c)[\omega - \mathrm{i}\omega_d(\boldsymbol{k})]. \qquad (4.15)$$

As mentioned at the beginning of this section, such a result is expected. No production and dissipation process being considered, a dynamical effect, in terms of energy exchange among the (random) component, would be not compatible with the above assumptions.

5. A particular example: wave field with zero bandwidth

We will illustrate the above analysis in the simplest case where the wave field oscillations are assumed to be associated with a single wavenumber $\boldsymbol{k}_0$ and a single frequency ω_0. In this case, $N_1(\boldsymbol{k}', \omega')$ and $B_1(\boldsymbol{k}', \omega')$ are written

$$N_1(\boldsymbol{k}', \omega') = \tfrac{1}{2}A_{\eta,0}[\delta(\boldsymbol{k}' - \boldsymbol{k}_0)\delta(\omega' - \omega_0) - \delta(\boldsymbol{k}' + \boldsymbol{k}_0)\delta(\omega' + \omega_0)], \tag{5.1}$$

$$B_1(\boldsymbol{k}', \omega') = \tfrac{1}{2}A_{\phi,0}[\delta(\boldsymbol{k}' - \boldsymbol{k}_0)\delta(\omega' - \omega_0) + \delta(\boldsymbol{k}' + \boldsymbol{k}_0)\delta(\omega' + \omega_0)], \tag{5.2}$$

where δ represents the Dirac delta function. Accounting for the transformations from the Fourier space to the physical space, simple algebra shows that the water deflection level corresponding to (5.1) is:

$$\eta(\boldsymbol{x}, t) = A_{\eta,0} \int_{\boldsymbol{x}_1} \int_{t_1} \cos\left[\boldsymbol{k}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_1) + \omega_0(t - t_1)\right] H^{(1)}(\boldsymbol{x}_1, t_1)\, \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}t_1. \tag{5.3}$$

A similar expression is obtained for the velocity potential.

The various statistical quantities defined previously become as follows:

$$\nu_{1N}(\boldsymbol{k}_0) = 0, \tag{5.4}$$

$$\omega_c = \tfrac{1}{2}\mathrm{i}k_0^3 \left[\exp\left(\tfrac{1}{2}\epsilon_0^2\right) - 1\right] A_{\eta,0}A_{\phi,0}^*, \tag{5.5}$$

$$\boldsymbol{V}_c = 0, \tag{5.6}$$

$$\omega_d(\boldsymbol{k}_0) = \mathrm{i}k_0^3 \exp\left(\tfrac{3}{4}\epsilon_0^2\right) A_{\eta,0}A_{\phi,0}^*, \tag{5.7}$$

$$\gamma_1 = 0, \tag{5.8}$$

$$\gamma_2 = k_0^3 \exp\left(\epsilon_0^2\right) |A_{\phi,0}|^2, \tag{5.9}$$

$$\gamma_3 = -k_0^5 \exp\left(\epsilon_0^2\right) (A_{\eta,0}A_{\phi,0}^*)^2, \tag{5.10}$$

with

$$\epsilon_0 = k_0|A_{\eta,0}|. \tag{5.11}$$

The actual forms of the kernels are $\nu_{1N}(\boldsymbol{k})$ for any $\boldsymbol{k}$. This is expected as this quantity represents here the change in the wavenumber through interactions respectively with the modes $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$. In the average, these interactions have opposite effects. The null values of $\boldsymbol{V}_c$ and γ_1 can also be explained that way.

The kinematical and dynamical boundary conditions (4.3) and (4.4) now are writen respectively:

$$k_0 \exp\left(\tfrac{1}{4}\epsilon_0^2\right) A_{\phi,0} + \mathrm{i}\left\{\omega_0 + \tfrac{1}{2}\mathrm{i}k_0^3\left[\exp\left(\tfrac{1}{4}\epsilon_0^2\right) - 1\right] A_{\eta,0}A_{\phi,0}^*\right\} A_{\eta,0} = 0, \tag{5.12}$$

$$-\mathrm{i}\left[\omega_0 \exp\left(\tfrac{1}{4}\epsilon_0^2\right) + \mathrm{i}k_0^3 \exp\left(\tfrac{3}{8}\epsilon_0^2\right) A_{\eta,0}A_{\phi,0}^*\right] A_{\phi,0} + \left[g + k_0^3 \exp\left(\epsilon_0^2\right) |A_{\phi,0}|^2 - k_0^5 \exp\left(\tfrac{1}{4}\epsilon_0^2\right) (A_{\eta,0}A_{\phi,0}^*)^2\right] A_{\eta,0} = 0. \tag{5.13}$$

Writing the complex amplitudes as

$$A_{\eta,0} = n \exp(i\theta_\eta), \tag{5.14}$$

$$A_{\phi,0} = b \exp(\mathrm{i}\theta_\phi), \tag{5.15}$$

equation (5.12) yields

$$b = -\frac{\mathrm{i}\omega \exp(\mathrm{i}\theta)}{k_0\left\{\exp\left(\tfrac{1}{4}\epsilon_0^2\right) - \tfrac{1}{2}\epsilon_0^2\left[\exp\left(\tfrac{1}{4}\epsilon_0^2\right) - 1\right] \exp(2\mathrm{i}\theta)\right\}} \tag{5.16}$$

with

$$\theta = \theta_\eta - \theta_\phi. \tag{5.17}$$

Simple algebra shows that b and n being real and positive numbers requires that $\theta = \frac{1}{2}\pi$. Then, in the average, the velocity potential lags the water surface deflection level by $\frac{1}{2}\pi$. In fact, owing to the forms of the basic equations (4.3) and (4.4), this result is general and is not limited to the particular spectral distribution we have chosen (see Joelson 1997). As a consequence,

$$b = -\frac{\mathrm{i}\omega_0 n}{k_0 F(m_0)} \tag{5.18}$$

with

$$F(m_0) = (1 + m_0)\exp\left(\tfrac{1}{2}m_0\right) - m_0 \tag{5.19}$$

and $m_0 = \frac{1}{2}\epsilon_0$ is the dimensionless variance of the wave field.

Finally, taking into account (5.14), (5.15) and (5.18), equation (5.13) yields

$$\omega_0 = (gk_0)^{1/2}\frac{F(m_0)}{\left[F(m_0)\exp\left(\frac{1}{2}m_0\right) - 4m_0(1+m_0)\exp(2m_0)\right]^{1/2}}. \tag{5.20}$$

Then,

$$\omega_0 = \omega_0(k_0, m_0) \tag{5.21}$$

which clearly expresses a nonlinear dispersion.

The relation (5.20) yields an important result as a consequence of the condition of stationarity. This condition requires real values of ω_0. This is found to be realized only for values of the dimensionless variance m_0 less than about 0.1797. The corresponding dimensionless standard deviation is about 0.4240. Note than small errors in the evaluation of the additional accelerations cumulated to yield to a wrong estimate (about 0.25) of this limiting value in Joelson & Ramamonjiarisoa (1999).

Figure 2 displays the variation with m_0 of the function $G(m_0)$ defined as

$$G(m_0) = \frac{F(m_0)}{\left[F(m_0)\exp\left(\frac{1}{2}m_0\right) - 4m_0(1+m_0)\exp(2m_0)\right]^{1/2}}. \tag{5.22}$$

For very small values of the variance, the dispersion relation (5.20) reduces to $\omega_0 \approx (gk_0)^{1/2}$, that is the dispersion relation for a linear deterministic wavefield.

More generally, as seen on figure 2, the frequency and thus the wave celerity increase with the wave field variance, first slowly and then very abruptly when the above critical variance is approached.

6. A non-Gaussian, second-order model

As mentioned previously, if the Wiener–Hermite expansion is carried up to order higher than unity, the procedure will define a non-Gaussian wave field. Unfortunately, on the basis of the exact boundary conditions (3.9) and (3.10), it has been seen that even for the first-order expansion, the algebra is already long and very complicated. The complexity increases very strongly for higher-order expansions. For this reason the following development using a second-order expansion will be based on approximate forms of the hydrodynamic boundary conditions. In the field of random motion of nonlinear systems, mathematical justification of such a perturbation procedure can be found in Crandall (1963).

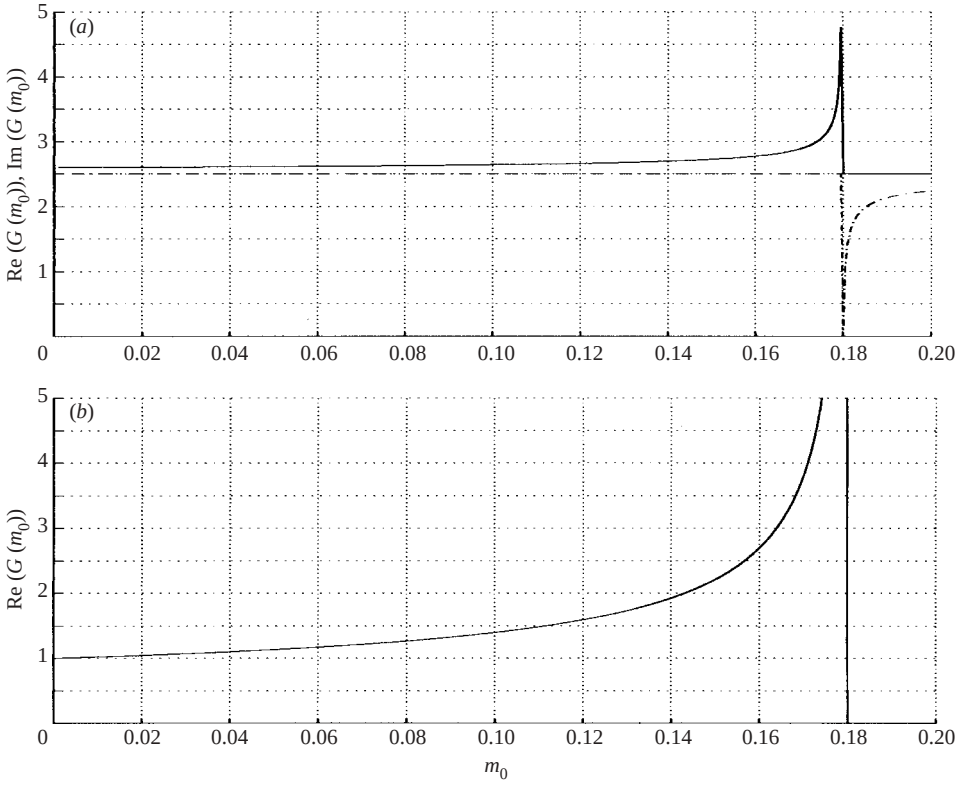

FIGURE 2. (*a*) Variations of the real (—) and the imaginary (—·—) parts of G with respect to the dimensionless variance m_0. (*b*) Close-up view of the variation of the real part of G.

The second-order model is defined by limiting the sums in the boundary conditions (3.9) and (3.10) to their respective first terms. Then, we have

the kinematic condition

$$k\,\mathrm{d}B(\boldsymbol{k},\omega)+\mathrm{i}\omega\,\mathrm{d}N(\boldsymbol{k},\omega) = -\int_{\boldsymbol{k}'}\int_{\omega'}[|\boldsymbol{k}-\boldsymbol{k}'|^2+\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')]\,\mathrm{d}B(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\mathrm{d}N(\boldsymbol{k}',\omega'), \tag{6.1}$$

the dynamic condition

$$\mathrm{i}\omega\,\mathrm{d}B(\boldsymbol{k},\omega)-g\,\mathrm{d}N(\boldsymbol{k},\omega) = -\mathrm{i}\int_{\boldsymbol{k}'}\int_{\omega'}(\omega-\omega')|\boldsymbol{k}-\boldsymbol{k}'|\,\mathrm{d}B(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\mathrm{d}N(\boldsymbol{k}',\omega') + \frac{1}{2}\int_{\boldsymbol{k}'}\int_{\omega'}[k'|\boldsymbol{k}-\boldsymbol{k}'|-\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')]\,\mathrm{d}B(\boldsymbol{k}',\omega')\,\mathrm{d}B(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'). \tag{6.2}$$

To derive the kernel equations in the Wiener–Hermite expansions, the random amplitudes are now written up to the second order as

$$\mathrm{d}N(\boldsymbol{k},\omega) = \eta^{(1)}(\boldsymbol{k},\omega)\tilde{H}^{(1)}(\boldsymbol{k},\omega)+(2\pi)^3\int_{\boldsymbol{p}}\int_{q}\eta^{(2)}(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q) \times\tilde{H}^{(2)}(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}q, \tag{6.3}$$

$$\mathrm{d}B(\boldsymbol{k},\omega)=\phi^{(1)}(\boldsymbol{k},\omega)\tilde{H}^{(1)}(\boldsymbol{k},\omega)+(2\pi)^3\int_{\boldsymbol{p}}\int_{q}\phi^{(2)}(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q)\times\tilde{H}^{(2)}(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}q \quad (6.4)$$

where $\eta^{(1)}$ and $\phi^{(1)}$ are the first-order kernels while $\eta^{(2)}$ and $\phi^{(2)}$ are the second-order kernels. The mathematical procedure to derive the kernel equations is again very tedious and lengthy. We will limit ourselves here to some indications concerning the procedure, for which details can be found in Joelson (1997).

To obtain the equations for the first-order kernels, the above expansions are introduced in equations (6.1) and (6.2). The equations so derived are multiplied by $\tilde{H}^{(1)}(-\boldsymbol{k},-\omega)$ and the ensemble averages taken. Accounting for the properties of the Wiener–Hermite set, especially the orthogonality property, and keeping only the first-order terms yields

$$k\phi^{(1)}(\boldsymbol{k},\omega)+\mathrm{i}\omega\eta^{(1)}(\boldsymbol{k},\omega)=0, \quad (6.5)$$

$$\mathrm{i}\omega\phi^{(1)}(\boldsymbol{k},\omega)-g\eta^{(1)}(\boldsymbol{k},\omega)=0. \quad (6.6)$$

The derivation of the second-order kernel equations proceeds in the same way: The expansions (6.3) and (6.4) are introduced in (6.1) and (6.2) but the equations so derived are now multiplied by $\tilde{H}^{(2)}(-\boldsymbol{k}_1,-\omega_1,-\boldsymbol{k}_2,-\omega_2)$ and the ensemble average taken. To illustrate the procedure, let us consider the terms which correspond to the integral $\int_{\boldsymbol{k}'}\int_{\omega'}[\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')]\,\mathrm{d}B(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\mathrm{d}N(\boldsymbol{k}',\omega')$ in the kinematic equation:

They are of three types:

(*a*) $\eta^{(1)}\phi^{(2)}$ or $\eta^{(2)}\phi^{(1)}$ multiplied by $\langle\tilde{H}^{(1)}\tilde{H}^{(2)}\tilde{H}^{(2)}\rangle$: this average is known to vanish;

(*b*) those of order higher than 2 and thus neglected;

(*c*)

$$\eta^{(1)}(\boldsymbol{k}',\omega')\phi^{(1)}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\big\langle\tilde{H}^{(1)}(\boldsymbol{k}',\omega')\tilde{H}^{(1)}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\tilde{H}^{(2)}(-\boldsymbol{k}_1,-\omega_1,-\boldsymbol{k}_2,-\omega_2)\big\rangle$$
$$=\eta^{(1)}(\boldsymbol{k}',\omega')\phi^{(1)}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\big\langle\tilde{H}^{(2)}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega',\boldsymbol{k}',\omega')\tilde{H}^{(2)}(-\boldsymbol{k}_1,-\omega_1,-\boldsymbol{k}_2,-\omega_2)\big\rangle.$$

But,

$$\big\langle\tilde{H}^{(2)}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega',\boldsymbol{k}',\omega')\tilde{H}^{(2)}(-\boldsymbol{k}_1,-\omega_1,-\boldsymbol{k}_2,-\omega_2)\big\rangle=\left(\frac{1}{2\pi}\right)^6[\delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}_1,\omega-\omega'-\omega_1)\delta(\boldsymbol{k}-\boldsymbol{k}_2,\omega-\omega_2)+\delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}_2,\omega-\omega'-\omega_2)\delta(\boldsymbol{k}'-\boldsymbol{k}_1,\omega'-\omega_1)].$$

Then, we have two possibilities

(i) $\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2$ and $\boldsymbol{k}'=\boldsymbol{k}_2$,

(ii) $\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2$ and $\boldsymbol{k}'=\boldsymbol{k}_1$;

case (i) yields $(1/2\pi)^6\eta^{(1)}(\boldsymbol{k}_1,\omega_1)\phi^{(1)}(\boldsymbol{k}_2,\omega_2)$,
case (ii) yields $(1/2\pi)^6\eta^{(1)}(\boldsymbol{k}_2,\omega_2)\phi^{(1)}(\boldsymbol{k}_1,\omega_1)$.

Finally, the above integral yields the term $2\boldsymbol{k}_1\cdot\boldsymbol{k}_2(1/2\pi)^6\eta^{(1)}(\boldsymbol{k}_1,\omega_1)\phi^{(1)}(\boldsymbol{k}_2,\omega_2)$ in the kernel equations. The others terms can be obtained that way.

The equations finally become

$$|\boldsymbol{k}_1+\boldsymbol{k}_2|\phi^{(2)}(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)+\mathrm{i}(\omega_1+\omega_2)\eta^{(2)}(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)=\left(\frac{1}{2\pi}\right)^3(\boldsymbol{k}_1+\boldsymbol{k}_2)^2\big[\eta^{(1)}(\boldsymbol{k}_1,\omega_1)\phi^{(1)}(\boldsymbol{k}_2,\omega_2)\big] \quad (6.7)$$

and

$$\mathrm{i}(\omega_1+\omega_2)\phi^{(2)}(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)-g\eta^{(2)}(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)=\left(\frac{1}{2\pi}\right)^3$$
$$\times\left[(\omega_1\boldsymbol{k}_1+\omega_2\boldsymbol{k}_2)\phi^{(1)}(\boldsymbol{k}_1,\omega_1)\eta^{(1)}(\boldsymbol{k}_2,\omega_2)-(k_1k_2-\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\phi^{(1)}(\boldsymbol{k}_1,\omega_1)\phi^{(1)}(\boldsymbol{k}_2,\omega_2)\right]. \quad (6.8)$$

Solutions

The equations for the first-order kernels (6.5) and (6.6) have an immediate solution in terms of a dispersion relation and a relationship between the first kernels, namely

$$\omega^2=gk, \quad (6.9)$$

$$\phi^{(1)}=-\mathrm{i}\frac{\omega}{k}\eta^{(1)}. \quad (6.10)$$

Then equations (6.7) and (6.8) allow us to write the second-order kernels in terms of the first-order kernels. For the application which follow, we will be interested in the random variation of water surface elevation. The corresponding second-order kernel is

$$\eta^{(2)}(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)=\left(\frac{1}{2\pi}\right)^3 M(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)\eta^{(1)}(\boldsymbol{k}_1,\omega_1)\eta^{(1)}(\boldsymbol{k}_2,\omega_2) \quad (6.11)$$

in which

$$M(\boldsymbol{k}_1,\omega_1,\boldsymbol{k}_2,\omega_2)=\frac{(\omega_1+\omega_2)F\dfrac{\omega_2}{k_2}-K\dfrac{\omega_1}{k_1}\left(G+\dfrac{1}{2}H\dfrac{\omega_2}{k_2}\right)}{(\omega_1+\omega_2)^2-gK} \quad (6.12)$$

with $F=(\boldsymbol{k}_1+\boldsymbol{k}_2)^2$; $G=(\omega_1\boldsymbol{k}_1+\omega_2\boldsymbol{k}_2)$; $H=k_1k_2-\boldsymbol{k}_1\cdot\boldsymbol{k}_2$ and $K=|\boldsymbol{k}_1+\boldsymbol{k}_2|$.

7. Example: zero bandwidth second-order wave field

As in § 5, let us consider again the specific case where the first-order kernels correspond to a Dirac delta function, namely

$$\eta^{(1)}(\boldsymbol{k}_i,\omega_i)=\tfrac{1}{2}A_0[\delta(\boldsymbol{k}_i-\boldsymbol{k}_0,\omega_i-\omega_0)-\delta(\boldsymbol{k}_i+\boldsymbol{k}_0,\omega_i+\omega_0)] \quad (7.1)$$

with $i=(1,2)$ and A_0 and $\boldsymbol{k}_0$ are constant (complex) amplitude and wavenumber; the notation $\delta(\boldsymbol{k},\omega)=\delta(\boldsymbol{k})\delta(\omega)$ is used for simplicity. Taking into account (6.3) and (6.11), the expression for the random amplitude of the water surface deflection level is now of the form

$$\mathrm{d}N(\boldsymbol{k},\omega)=\tfrac{1}{2}A_0[\delta(\boldsymbol{k}-\boldsymbol{k}_0,\omega-\omega_0)-\delta(\boldsymbol{k}+\boldsymbol{k}_0,\omega+\omega_0)]\tilde{H}^{(1)}(\boldsymbol{k},\omega)$$
$$+\frac{A_0^2}{4}\int_{\boldsymbol{p}}\int_q M(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q)[\delta(\boldsymbol{k}-\boldsymbol{k}_0,\omega-\omega_0)\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{k}_0,\omega-q-\omega_0)$$
$$+\delta(\boldsymbol{k}+\boldsymbol{k}_0,\omega+\omega_0)\delta(\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{k}_0,\omega-q+\omega_0)]\tilde{H}^{(2)}(\boldsymbol{p},q,\boldsymbol{k}-\boldsymbol{p},\omega-q)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}q \quad (7.2)$$

which immediately gives

$$\mathrm{d}N(\boldsymbol{k},\omega)=\tfrac{1}{2}A_0[\delta(\boldsymbol{k}-\boldsymbol{k}_0,\omega-\omega_0)-\delta(\boldsymbol{k}+\boldsymbol{k}_0,\omega+\omega_0)]\tilde{H}^{(1)}(\boldsymbol{k},\omega)$$
$$+\frac{A_0^2}{4}\big[M(\boldsymbol{k}_0,\omega_0,\boldsymbol{k}-\boldsymbol{k}_0,\omega-\omega_0)\delta(\boldsymbol{k}-2\boldsymbol{k}_0,\omega-2\omega_0)\tilde{H}^{(2)}(\boldsymbol{k}_0,\omega_0,\boldsymbol{k}-\boldsymbol{k}_0,\omega-\omega_0)$$
$$+M(-\boldsymbol{k}_0,-\omega_0,\boldsymbol{k}+\boldsymbol{k}_0,\omega+\omega_0)\delta(\boldsymbol{k}+2\boldsymbol{k}_0,\omega+2\omega_0)\tilde{H}^{(2)}(-\boldsymbol{k}_0,-\omega_0,\boldsymbol{k}$$
$$+\boldsymbol{k}_0,\omega+\omega_0)\big]. \quad (7.3)$$

From (7.3) the variations comprise two components: a first-order, Gaussian component, $\eta_1(\boldsymbol{x}, t)$, and a second-order, non-Gaussian component, $\eta_2(\boldsymbol{x}, t)$:

$$\eta(\boldsymbol{x}, t) = \eta_1(\boldsymbol{x}, t) + \eta_2(\boldsymbol{x}, t). \tag{7.4}$$

Clearly, the second-order component is of wavenumber and frequency $(2\boldsymbol{k}_0, 2\omega_0)$ respectively. Then, the second-order nonlinear effect results in the generation of the second harmonic component of the primary wave.

As in § 5, transformations from the Fourier space to the physical space yield

$$\eta_1(\boldsymbol{x}, t) = A_{\eta,0} \int_{\boldsymbol{x}_1} \int_{t_1} \cos\left[\boldsymbol{k}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_1) + \omega_0(t - t_1)\right] H^{(1)}(\boldsymbol{x}_1, t_1)\, \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}t_1, \tag{7.5}$$

$$\eta_2(\boldsymbol{x}, t) = \frac{A_{\eta,0}}{2} M_0 \int_{\boldsymbol{x}_1} \int_{t_1} \int_{\boldsymbol{x}_2} \int_{t_2} \cos[\boldsymbol{k}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_1 + \boldsymbol{x} - \boldsymbol{x}_2) + \omega_0(t - t_1 + t - t_2)] \times H^{(2)}(\boldsymbol{x}_1, t_1, \boldsymbol{x}_2, t_2)\, \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}t_1\, \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}t_2 \tag{7.6}$$

where, according to (6.12), $M_0 = M(\boldsymbol{k}_0, \omega_0, \boldsymbol{k}_0, \omega_0) = M(-\boldsymbol{k}_0, -\omega_0, -\boldsymbol{k}_0, -\omega_0)$.

We have

$$M_0 = \frac{2k_0\omega_0^2}{2\omega_0^2 - gk_0}. \tag{7.7}$$

But

$$\omega_0^2 = gk_0. \tag{7.8}$$

Thus, finally

$$M_0 = 2k_0. \tag{7.9}$$

The expressions (7.5) and (7.6) allow us, in principle, to construct realizations of $\eta(\boldsymbol{x}, t)$ although this construction is not a simple matter because it involves computations of multiple convolution integrals.

We will illustrate the random wave forms by considering the following simplified, only time-dependent, expressions:

$$\eta_1(t) = A_{\eta,0} \int_{t_1} \cos\left[\omega_0(t - t_1)\right] H^{(1)}(t_1)\, \mathrm{d}t_1, \tag{7.10}$$

$$\eta_2(t) = \frac{A_{\eta,0}}{2} M_0 \int_{t_1} \int_{t_2} \cos\left[\omega_0(t - t_1 + t - t_2)\right] H^{(2)}(t_1, t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2. \tag{7.11}$$

Figure 3 displays samples of $H^{(1)}(t)$, $\eta_1(t)$, $\eta_2(t)$ and $\eta(t) = \eta_1(t) + \eta_2(t)$. The remarkable fact, from figure 3, is that, compared to the first-order wave form, the second-order component makes the wave crests sharper and the wave trough flatter. This compares with the second-order effect in deterministic waves (Stokes waves) and is generally observed in experiments in laboratory facilities on wind-generated waves.

Simple algebra allows us to show that the variances of the first-order and the second-order components are respectively

$$\langle \eta_1(t)^2 \rangle = \frac{|A_0|^2}{2}, \tag{7.12}$$

$$\langle \eta_2(t)^2 \rangle = \frac{|A_0|^4}{8} {M_0}^2. \tag{7.13}$$

Then, with $M_0 = 2k_0$ and denoting $\epsilon_0 = |A_0|k_0$, a statistical wave steepness, we have

$$\langle {\eta_2}^2 \rangle = {\epsilon_0}^2 \langle {\eta_1}^2 \rangle.$$

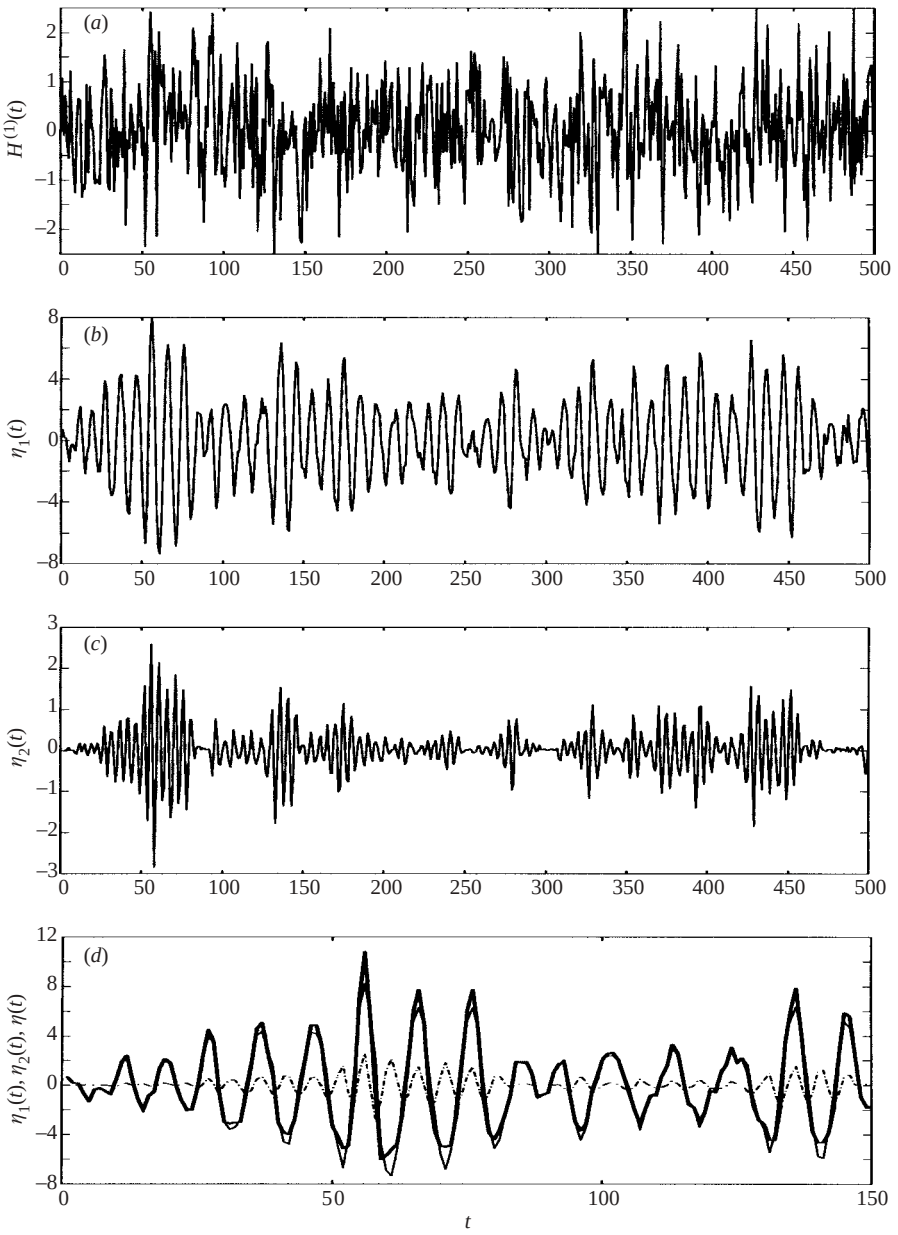

FIGURE 3. (*a*) Long sample of normalized 'time' evolution of $H^{(1)}(t)$. (*b*) Sample of normalized realization of the first order component, $\eta_1(t)$, of the water surface deflection level. (*c*) Sample of normalized realization of the second-order component, $\eta_2(t)$, of the water surface deflection. (*d*) Samples of normalized realizations of $\eta_1(t)$ (thin line), $\eta_2(t)$ (dashed line) and $\eta(t) = \eta_1(t) + \eta_2(t)$ (heavy line), with $\langle {\eta_2}^2 \rangle = {\epsilon_0}^2$ and $\epsilon_0 = 0.25$.

8. Discussions and conclusions

The first aim of the work presented in this paper was to carry out the mathematical formulation of random water surface waves using the Wiener–Hermite functional series expansion. Such a formulation yields equations for the kernels, which characterize the transformation from the Wiener set of elementary random

processes to the random water surface fields. The work is possibly the first attempt in that respect. Two types of stationary and homogeneous (in the horizontal plane) random fields were considered. In the first, no approximation is made on the basic hydrodynamic equation and nonlinear boundary conditions but the Wiener–Hermite expansion was limited to the first order. The corresponding water surface field is then Gaussian. As expected, the nonlinearity results in a nonlinear dispersion: the relationship between the wavenumber and the frequency depends upon the variances of the water surface deflection level and the velocity potential at the water surface as well as the upon the covariance of these two physical variables through additional accelerations and velocity at the surface. It is seen that, as expected in a Gaussian field, the wavenumber is referred to the standard deviation of the water surface deflection level. In the average, the velocity potential at the surface is found to lead the water deflection level by $\frac{1}{2}\pi$. This is reminiscent of the result from the deterministic case. In fact, the existence of additional velocity and accelerations at the surface is also well established in the deterministic case. However, much care must be taken in trying to relate our results with those found in the latter case as our results have to be taken in the statistical sense. No attempt was made to solve the coupled nonlinear integral equations for the kernels. Instead Dirac distributions were chosen *à priori* to illustrate the derived random field. In this case, all statistical terms characterizing the field can be determined analytically. The most remarkable fact is the existence of a limiting value of the variance implied by the stationarity. In terms of a statistical wave steepness, this value is of order 0.42. Beyond that value, the frequency takes an infinite negative value, representing an infinite damping of the wave amplitude. Quite clearly, this results from an infinite value of the additional accelerations at the water surface. If the variance is very small, the dispersion relation reduces to that of a linear deterministic wave field. Again, it is to be stressed that all our results are to be understood on statistical grounds.

In the second case, the hydrodynamic boundary conditions are limited to the second order and correspondingly, the Wiener–Hermite functional series expansion is also limited to the second order. Then, a non-Gaussian water surface field is defined. This 'double approximation' procedure is quite reminiscent of that of Tick (1959) but the use of the Wiener–Hermite expansion made here is original. As in the previous case, the kernel equations are derived and the case of a Dirac distribution treated as comprehensive illustrative example. As expected, the second-order effects result in the generation of the second harmonic of the fundamental, first-order wave component. Of main interest is that the ratio of the variances of the second-order and the fundamental component is equal to the square of the statistical wave steepness of the fundamental component. This compares quite well with the results commonly observed in laboratory experiments in which the frequency bandwidths of the wave fields is quite narrow (see e.g. Leykin *et al.* 1995). Realizations of the wave fields can be constructed in this second case. The similarity with experimental samples is again striking: the wave crests are sharpened and the wave troughs flattened due to the second-harmonic component.

This second-order model constitutes, in the stochastic field, the equivalent of the second-order Stokes wave of permanent form in the deterministic field. In view of the asymmetric waveforms, resembling that of Stokes waves, found in laboratory experiments on wind waves, results from the latter were often analysed with reference to these waves. This is clearly not justified from the mathematical viewpoint as wind waves are not of permanent form. The model could be a candidate for a more correct interpretation of the results. However, the model needs possibly to

be extended to account for the non-zero bandwidth. Also, in order to extend the validity of the model and to estimate the uncertainty involved in the truncation order of the development, a quantitative comparison between the result of the model and experiment data will be needed. One prospective task that we shall undertake concerns the analysis of laboratory wind wave field in which dynamic evolution was recently found to be "related with competitive deterministic and random features whose correct mathematical and physical formulation has to be found" (see Joelson *et al.* 2000).

According to this work, various fields of random water surface waves can be studied with the use of the Wiener–Hermite functional expansion. The main difficulty encountered arises from the long and complicated algebraic operations. The use of symbolic calculus as developed by Imamura, Meecham & Siegel (1965) may be of great help in that respect. Also, once the kernel equations are established, adapted numerical techniques of resolution such as those found in Xui *et al.* (2002) may be implemented. Among the various applications in the realm of physical oceanography, one topic worthy of investigation concerns the direct simulation of electromagnetic scattering by individual realizations of the random sea surface (National Research Council 1994) such as those constructed by our present model. Generalization of the present simulation to spatial and time varying cases will be required for such purpose.

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